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Asymptotic Formulas for the Zeros of the Meixner Polynomials

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The zeros of the Meixner polynomial $m_n(x; \beta, c)$ are real, distinct, and lie in $(0, \infty)$. Let $\alpha_{n,s}$ denote the sth zero of $m_n(n\alpha; \beta, c)$, counted from the right; and let $\bar{\alpha}_{n,s}$ denote the sth zero of $m_n(n\alpha; \beta, c)$, counted from the left. For each fixed s, asymptotic formulas are obtained for both $\alpha_{n,s}$ and $\bar{\alpha}_{n,s}$, as $n \to \infty$. © 1999 Academic Press

1. INTRODUCTION

One of the major topics in the study of orthogonal polynomials is investigating the properties of their zeros. While a considerable amount of literature already exists on the zeros of classical orthogonal polynomials of Hermite, Laguerre, and Jacobi (see, e.g., [12]), not much is known about the zeros of the non-classical polynomials such as Charlier $C_n^{(a)}(x)$, Meixner $m_n(x; \beta, c)$, Pollaczek $P_n(a, b; x)$, and Meixner–Pollaczek $M_n(x; \eta, \delta)$. An asymptotic formula for the zeros of the Pollaczek polynomial was first given by Novikoff [9] in 1954, but the problem of finding the second term in the asymptotic expansion of these zeros was settled only very recently; see [2, 5]. In 1992, Ismail and Li [7] presented an upper bound for the largest zero, and a lower bound for the smallest zero, of the Meixner–Pollaczek polynomial, and also gave an upper bound for the largest zero of the Meixner polynomial. More recently, Chen and Ismail [3] obtained bounds for these zeros which are sharp for large n; see also [6]. As regards the Charlier polynomial, practically nothing is known about its zeros.

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In this paper, we are concerned with the zeros of the Meixner polynomial $m_n(n\alpha; \beta, c)$, where $0 < \alpha < \infty$. First, let $\alpha_{n,s}$ denote its sth zero, counted in descending order,

$$0 < \alpha_{n,n} < \alpha_{n,n-1} < \cdots < \alpha_{n,2} < \alpha_{n,1} < \infty. \tag{1.1}$$

We shall show that for each fixed s we have

$$\alpha_{n,s} = \frac{1+\sqrt{c}}{1-\sqrt{c}} + \frac{c^{1/6}(1+\sqrt{c})^{1/3}}{1-\sqrt{c}} \frac{a_s}{n^{2/3}} + O\left(\frac{1}{n}\right) \tag{1.2}$$

as $n \to \infty$, where a_s is the sth negative zero of the Airy function $Ai(x)$. Next, let $\bar{\alpha}_{n,s}$ denote the sth zero of $m_n(n\alpha; \beta, c)$, counted in ascending order,

$$0 < \bar{\alpha}_{n,1} < \bar{\alpha}_{n,2} < \cdots < \bar{\alpha}_{n,n-1} < \bar{\alpha}_{n,n} < \infty. \tag{1.3}$$

Clearly, we have

$$\bar{\alpha}_{n,s} = \alpha_{n,n-s+1}, \qquad s = 1, ..., n. \tag{1.4}$$

We shall also show that for each s,

$$\bar{\alpha}_{n,s} = \frac{s-1}{n} + \text{an exponentially small error}. \tag{1.5}$$

Numerical values of $\bar{\alpha}_{n,s}$ (see Tables 1–4) match closely the approximate values obtained from (1.5).

2. BEHAVIOR OF THE LARGE ZEROS

To derive the asymptotic formula in (1.2), we first recall a uniform asymptotic approximation of $m_n(n\alpha; \beta, c)$ given in [8]. Let $U(a, x)$ denote the Weber parabolic cylinder function [10, p. 687], and put

$$V_n(x) = e^{x^2/4} U(-n - \tfrac{1}{2}, x). \tag{2.1}$$

Note that $V_n(x)$ can be expressed in terms of the Hermite polynomial; in fact, we have $V_n(x) = 2^{-n/2} H_n(x/\sqrt{2})$; see [11, p. 259]. Consider the functions

$$G(w, \alpha, c) = \alpha \log\left(1 - \frac{w}{c}\right) - \alpha \log(1 - w) - \log(-w) \tag{2.2}$$

and

$$\Psi(u, \eta) = -\log u + \eta u - \frac{u^2}{2}. \tag{2.3}$$

Clearly, $G(w, \alpha, c)$ and $\Psi(u, \eta)$ have the saddle points

$$w_{\pm} = \frac{1 + c + ac - \alpha + \sqrt{(1 + c + \alpha c - \alpha)^2 - 4c}}{2} \tag{2.4}$$

and

$$u_{\pm} = \frac{\eta \pm \sqrt{\eta^2 - 4}}{2}, \tag{2.5}$$

respectively. In [8], it has been proved that the system of nonlinear equations

$$\begin{cases} G(w_+, \alpha, c) = \Psi(u_-, \eta) + \gamma, \\ G(w_-, \alpha, c) = \Psi(u_+, \eta) + \gamma \end{cases} \tag{2.6}$$

has a unique solution (η, γ) for $\alpha \geqslant 1 - b$, where

$$0 < b < \min \left\{ \frac{1}{2}, \frac{2\sqrt{c}}{1 + \sqrt{c}} \right\},$$

and furthermore, $2 \leqslant \eta < \infty$ when $\alpha_+ \leqslant \alpha < \infty$ and $-2 + \delta \leqslant \eta < 2$ when $1 - b \leqslant \alpha < \alpha_+$, where

$$\alpha_+ = \frac{1 + \sqrt{c}}{1 - \sqrt{c}} \tag{2.7}$$

and δ is a positive number depending on b and c. Also, it has been shown that the function $u = u(w, \alpha)$ defined by

$$G(w, \alpha, c) = \Psi(u, \eta) + \gamma \tag{2.8}$$

is one-to-one and analytic in the w-plane except possibly on the cut along the positive real axis, and is bounded and analytic near the origin. Set

$$h(u) = \frac{u}{w(1 - w)^{\beta}} \frac{dw}{du}. \tag{2.9}$$

From (2.7), we have

$$h(u) = \frac{(c - w)(u - u_+)(u - u_-)}{(w - w_+)(w - w_-)(1 - w)^{\beta - 1}}. \tag{2.10}$$

One of the two major results in [8] is the uniform asymptotic approximation

$$m_n(n\alpha;\beta,c)=(-1)^n\, n^{n/2}e^{n\gamma}\left[V_n(\eta\sqrt{n})\,a_0+\frac{1}{\sqrt{n}}\,V_n'(\eta\sqrt{n})\,b_0+\varepsilon_1\right],\quad (2.11)$$

where

$$|\varepsilon_1|\leqslant\frac{M_1}{n}\,|V_n(\eta\sqrt{n})|+\frac{N_1}{n^{3/2}}\,|V_n'(\eta\sqrt{n})|.\quad (2.12)$$

The constants M_1 and N_1 are independent of n and α for $\alpha\in[1-b,M]$, and M can be any fixed large number. The coefficients a_0 and b_0 are given explicitly by

$$a_0=\frac{u_-h(u_+)-u_+h(u_-)}{u_--u_+},\quad (2.13)$$

$$b_0=\frac{h(u_-)-h(u_+)}{u_--u_+}.\quad (2.14)$$

From the asymptotic behavior of the parabolic cylinder function given by Olver [10], it is readily seen that

$$V_n(\eta\sqrt{n})\,a_0+\frac{1}{\sqrt{n}}\,V_n'(\eta\sqrt{n})\,b_0$$

has no zero when $\eta\geqslant 2$ and n is sufficiently large; see also the asymptotic formulas of $V_n(\eta\sqrt{n})$ and $V_n'(\eta\sqrt{n})/\sqrt{n}$ given in [8, Section 6]. (Since $\eta\geqslant 2$ corresponds to $\alpha\geqslant\alpha_+$, we can conclude that $m_n(n\alpha;\beta,c)$ has no zero when $\alpha\geqslant\alpha_+$ and n is sufficiently large; cf. [7, Theorem 6].) We may therefore restrict ourselves to the case $\eta<2$. Let us now consider the behavior of $V_n(\eta\sqrt{n})$ and $V_n'(\eta\sqrt{n})$ when $n^{2/3}(\eta-2)$ is bounded and does not tend to zero; i.e., there exist positive numbers ρ and δ such that $-\rho<n^{2/3}(\eta-2)<-\delta$. It will be seen later in this section that it suffices to establish (1.2) under these restrictions. From [10, pp. 152–153], we have

$$U\left(-\frac{1}{2}\mu^2,\mu t\sqrt{2}\right)=2\sqrt{\pi}\,\mu^{1/3}g(\mu)\left(\frac{\zeta}{t^2-1}\right)^{1/4}$$
$$\times\left[Ai(\mu^{4/3}\zeta)+Ai'(\mu^{4/3}\zeta)\,\mu^{-8/3}B_0(\zeta)\right]\cdot\left[1+O(\mu^{-4})\right]\quad (2.15)$$

and

$$U'\left(-\frac{1}{2}\mu^2, \mu t\sqrt{2}\right) = \sqrt{2\pi}\,\mu^{2/3} g(\mu)\left(\frac{\zeta}{t^2-1}\right)^{-1/4}$$
$$\times [Ai'(\mu^{4/3}\zeta) + Ai(\mu^{4/3}\zeta)\,\mu^{-4/3} C_0(\zeta)] \cdot [1 + O(\mu^{-4})] \tag{2.16}$$

as $\mu \to +\infty$, uniformly with respect to $t \geqslant -1+\sigma > -1$ (see [10, p. 158]), where

$$\tfrac{2}{3}\zeta^{3/2} = \tfrac{1}{2} t\sqrt{t^2-1} - \tfrac{1}{2}\log(t + \sqrt{t^2-1}), \tag{2.17}$$

$$g(\mu) = 2^{-(1/4)\mu^2 - (1/4)} e^{-(1/4)\mu^2} \mu^{(1/2)\mu^2 - (1/2)} [1 + O(\mu^{-2})], \tag{2.18}$$

$$\zeta^{1/2} B_0(\zeta) = -\tfrac{1}{24}(t^3 - 6t)(t^2-1)^{-3/2} - \tfrac{5}{48}\zeta^{-3/2}, \tag{2.19}$$

$$\zeta^{-1/2} C_0(\zeta) = -\tfrac{1}{24}(t^3 + 6t)(t^2-1)^{-3/2} + \tfrac{7}{48}\zeta^{-3/2}. \tag{2.20}$$

From (2.19) and (2.20), it can be shown that

$$B_0(\zeta) \sim -\tfrac{9}{280} 2^{1/3} \qquad \text{and} \qquad C_0(\zeta) \sim -\tfrac{1}{20} 2^{2/3}, \qquad \text{as} \quad t \to 1^+. \tag{2.21}$$

Let

$$\mu = \sqrt{2n+1} \qquad \text{and} \qquad t = \eta\sqrt{\frac{n}{4n+2}}. \tag{2.22}$$

It is easily verified that

$$t - 1 = \tfrac{1}{2}(\eta - 2) + O(n^{-1}), \qquad \text{as} \quad t \to 1^+. \tag{2.23}$$

Since $n^{2/3}(\eta - 2)$ is bounded, it follows that

$$t - 1 = O(n^{-2/3}), \qquad \text{as} \quad t \to 1^+. \tag{2.24}$$

Combining (2.17), (2.23), and (2.24) gives

$$\zeta = 2^{1/3}(t-1) + O((t-1)^2) = O(n^{-2/3}), \tag{2.25}$$

$$\left(\frac{\zeta}{t^2-1}\right)^{1/4} = 2^{-1/6} + O(n^{-2/3}), \tag{2.26}$$

and

$$\mu^{4/3}\zeta = n^{2/3}(\eta - 2) + O(n^{-1/3}). \tag{2.27}$$

Substituting (2.26) and (2.27) into (2.15) and (2.16), we obtain

$$U(-\tfrac{1}{2}\mu^2, \mu t\sqrt{2}) = \sqrt{2\pi}\, e^{-n/2} n^{(n/2)+(1/6)} [Ai(n^{2/3}(\eta-2)) + O(n^{-1/3})], \tag{2.28}$$

$$U'(-\tfrac{1}{2}\mu^2, \mu t\sqrt{2}) = \sqrt{2\pi}\, e^{-n/2} n^{(n/2)+(1/3)} [Ai'(n^{2/3}(\eta-2)) + O(n^{-1/3})]. \tag{2.29}$$

From (2.1), it follows that

$$V_n(\eta\sqrt{n}) = \sqrt{2\pi}\, e^{n((\eta^2/4)-(1/2))} n^{(n/2)+(1/6)} [Ai(n^{2/3}(\eta-2)) + O(n^{-1/3})], \tag{2.30}$$

$$\frac{1}{\sqrt{n}} V_n'(\eta\sqrt{n}) = \sqrt{2\pi}\, e^{n((\eta^2/4)-(1/2))} n^{(n/2)+(1/6)} \left[Ai(n^{2/3}(\eta-2)) \frac{\eta}{2} + O(n^{-1/3}) \right], \tag{2.31}$$

as $n \to \infty$ and $n^{2/3}(\eta-2)$ bounded ($\eta < 2$). We may therefore rewrite (2.11) in the form

$$m_n(n\alpha; \beta, c) = (-1)^n\, n^{n+(1/6)} e^{n\gamma} \sqrt{2\pi}\, e^{n((\eta^2/4)-(1/2))} \times \left[Ai(n^{2/3}(\eta-2)) \left(a_0 + \frac{\eta}{2} b_0 \right) + \varepsilon_2 \right], \tag{2.32}$$

where by (2.12)

$$\varepsilon_2 = O(n^{-1/3}) + O(n^{-1}) = O(n^{-1/3}). \tag{2.33}$$

Note that from (2.5), we have

$$a_0 + \frac{\eta}{2} b_0 \sim a_0 + \left(\frac{\eta}{2} - \frac{\sqrt{\eta^2-4}}{2} \right) b_0 = a_0 + b_0 u_-, \qquad \text{as} \quad \eta \to 2^-.$$

Hence, on account of (2.13) and (2.14), we get

$$a_0 + \frac{\eta}{2} b_0 \sim \frac{u_- h(u_+) - u_+ h(u_-)}{u_- - u_+} + \frac{h(u_-) - h(u_+)}{u_- - u_+} u_- = h(u_-),$$

which in turn yields

$$a_0 + \frac{\eta}{2} b_0 \sim c^{-1/6} (1 + \sqrt{c})^{(2/3)-\beta} \tag{2.34}$$

by using (2.9). It is now evident that $a_0 + (\eta/2)\, b_0 \not\to 0$. Therefore, (2.32) can be further rewritten as

$$m_n(n\alpha;\, \beta, c) = (-1)^n\, n^{n+(1/6)} e^{n\gamma} \sqrt{2\pi}\, e^{n((\eta^2/4)-(1/2))} \times \left(a_0 + \frac{\eta}{2}\, b_0\right) [A_i(n^{2/3}(\eta - 2)) + \varepsilon_2]. \tag{2.35}$$

The asymptotic approximation in (1.2) is obtained from (2.35). To do this, we first recall the following result of Hethcote [4]:

LEMMA 1. *In the interval* $[a-\rho, a+\rho]$, *suppose* $f(t) = g(t) + \varepsilon(t)$, *where* $f(t)$ *is continuous,* $g(t)$ *is differentiable,* $g(a) = 0$, $m = \min|g'(t)| > 0$, *and*

$$E = \max|\varepsilon(t)| < \min\{|g(a-\rho)|, |g(a+\rho)|\}. \tag{2.36}$$

Then there exists a zero c *of* $f(t)$ *in the interval such that*

$$|c - a| \leqslant \frac{E}{m}. \tag{2.37}$$

We next study the behavior of η, when α is given by

$$\alpha = \alpha_+ + n^{-2/3} a = \frac{1+\sqrt{c}}{1-\sqrt{c}} + n^{-2/3} a, \tag{2.38}$$

where a is a bounded real parameter independent of n. Substituting (2.38) into (2.4), we have

$$w_+ = -\sqrt{c} + c^{1/4}(1-c)^{1/2}\, a^{1/2} n^{-1/3} + b_2 n^{-2/3} + b_3 n^{-1} + O(n^{-4/3}),$$

where b_2 and b_3 depend on a but are independent of n. With this value of w_+, (2.2) gives

$$G(w_+, \alpha, c) = -\frac{1+\alpha}{2} \log c - \frac{2}{3} (1+\sqrt{c})^{-1/2} \times c^{-1/4} (1-\sqrt{c})^{3/2}\, a^{3/2} n^{-1} + O(n^{-4/3}).$$

In the same manner, it follows that

$$G(w_-, \alpha, c) = -\frac{1+\alpha}{2} \log c + \frac{2}{3} (1+\sqrt{c})^{-1/2} \times c^{-1/4} (1-\sqrt{c})^{3/2}\, a^{3/2} n^{-1} + O(n^{-4/3}).$$

Subtracting the two equations in (2.6) and using the above values of $G(w_+, \alpha, c)$ and $G(w_-, \alpha, c)$, we obtain

$$-\frac{4}{3}(1+\sqrt{c})^{-1/2}\,c^{-1/4}(1-\sqrt{c})^{3/2}\,a^{3/2}n^{-1}+O(n^{-4/3})$$
$$=-2\left(\log\frac{\eta-\sqrt{\eta^2-4}}{2}+\frac{\eta\sqrt{\eta^2-4}}{4}\right). \tag{2.39}$$

On the other hand, inserting (2.22) into (2.17) yields

$$\frac{2}{3}\zeta^{3/2}=\frac{1}{2}\left[\log\frac{\eta-\sqrt{\eta^2-4}}{2}+\frac{\eta}{4}\sqrt{\eta^2-4}\right]+O(n^{-4/3}). \tag{2.40}$$

Coupling (2.39) and (2.40), we get

$$\frac{2}{3}\zeta^{3/2}=\frac{1}{3}(1+\sqrt{c})^{-1/2}\,c^{-1/4}(1-\sqrt{c})^{3/2}\,a^{3/2}n^{-1}+O(n^{-4/3}).$$

By (2.23) and (2.25),

$$\zeta=2^{-2/3}(\eta-2)+O(n^{-1})$$

and hence

$$n^{2/3}(\eta-2)=(1+\sqrt{c})^{-1/3}\,c^{-1/6}(1-\sqrt{c})\,a+O(n^{-1/3}), \tag{2.41}$$

when α is given by (2.38).

THEOREM 1. *Let $\alpha_{n,s}$ be the sth zero of the Meixner polynomial $m_n(n\alpha;\beta,c)$, arranged in descending order given in* (1.1), *and let a_s denote the sth negative zero of the Airy function $Ai(x)$. Then*

$$\alpha_{n,s}=\frac{1+\sqrt{c}}{1-\sqrt{c}}+\frac{c^{1/6}(1+\sqrt{c})^{1/3}}{1-\sqrt{c}}\frac{a_s}{n^{2/3}}+O\left(\frac{1}{n}\right) \tag{2.42}$$

as $n\to\infty$; i.e., formula (1.2) *holds.*

Proof. From (2.35) and (2.41), we have

$$m_n(n\alpha;\beta,c)=(-1)^n\,n^{n+(1/6)}e^{n\gamma}\sqrt{2\pi}\,e^{n((\eta^2/4)-(1/2))}\left(a_0+\frac{\eta}{2}b_0\right)$$
$$\times\{Ai[(1+\sqrt{c})^{-1/3}\,c^{-1/6}(1-\sqrt{c})\,a]+\tilde{\varepsilon}\}, \tag{2.43}$$

where

$$\tilde{\varepsilon} = O(n^{-1/3}) \qquad \text{uniformly in } \alpha \tag{2.44}$$

and α is given in (2.38). Let

$$f(\alpha) = (-1)^n \, n^{-n-(1/6)} e^{-n\gamma} (2\pi)^{-1/2} \, e^{n((1/2)-(\eta^2/4))} \left(a_0 + \frac{\eta}{2} \, b_0 \right)^{-1} m_n(n\alpha; \beta, c),$$

$$g(\alpha) = Ai \left[(1+\sqrt{c})^{-1/3} \, c^{-1/6} (1-\sqrt{c}) \left(\alpha - \frac{1+\sqrt{c}}{1-\sqrt{c}} \right) n^{2/3} \right],$$

and $\varepsilon(\alpha) = \tilde{\varepsilon}$. Then (2.43) becomes

$$f(\alpha) = g(\alpha) + \varepsilon(\alpha).$$

Note that for α given in (2.38), we have $\min |g'(\alpha)| \geqslant cn^{2/3}$ for some $c > 0$. Formula (2.42) now follows from Lemma 1. ∎

Remark 1. Recall that in the derivation of (2.30) and (2.31), we have restricted ourselves to the case in which $\eta < 2$ and $n^{2/3}(\eta - 2)$ is bounded away from zero. Since a_s is negative for all $s = 1, 2, ...$, it is readily seen from (2.41) that this restriction is reasonable.

3. UNIFORM ASYMPTOTICS NEAR $\alpha = 0$

To state the other major result in [8], we consider the functions

$$F(w, \alpha, c) = \alpha \log \left(1 - \frac{w}{c} \right) - \alpha \log(1-w) - \log w \tag{3.1}$$

and

$$\Phi(u, \eta) = -\alpha \log u + \eta u - \tfrac{1}{2} u^2. \tag{3.2}$$

The saddle points of $F(w, \alpha, c)$ and $\Phi(u, \eta)$ occur at

$$w_{\pm} = \frac{1 + c + \alpha c - \alpha \pm \sqrt{(1 + c + \alpha c - \alpha)^2 - 4c}}{2} \tag{3.3}$$

and

$$u_{\pm} = \frac{\eta \pm \sqrt{\eta^2 - 4\alpha}}{2}, \tag{3.4}$$

respectively. In [8], it has been established that the system of nonlinear equations

$$\begin{cases} F(w_+, \alpha, c) = \Phi(u_+, \eta) + \gamma \\ F(w_-, \alpha, c) = \Phi(u_-, \eta) + \gamma \end{cases} \tag{3.5}$$

has a unique solution (η, γ) for $0 < \alpha \leqslant 1 + a$, where

$$0 < a < \alpha_+ - 1 = \frac{2\sqrt{c}}{1 - \sqrt{c}},$$

and furthermore that $\eta < -2\sqrt{\alpha}$ when $0 < \alpha \leqslant \alpha_-$ and $-2\sqrt{\alpha} \leqslant \eta \leqslant 2\sqrt{\alpha} - \delta_0$ when $\alpha_- < \alpha \leqslant 1 + a$, where

$$\alpha_- = \frac{1 - \sqrt{c}}{1 + \sqrt{c}} \tag{3.6}$$

and δ is a positive number depending only on the values of a and c. It has also been proved that the function $u = u(w, \alpha)$ defined by

$$F(w, \alpha, c) = \Phi(u, \eta) + \gamma \tag{3.7}$$

is one-to-one and analytic in the w-plane except possibly on the cut along the negative real axis, and is bounded and analytic near the origin. As in (2.9) and (2.10), we again set

$$h(u) = \frac{u}{(1-u)^\beta} \frac{dw}{du} = \frac{(c - w)(u - u_+)(u - u_-)}{(w - w_+)(w - w_-)(1 - w)^{\beta - 1}}.$$

The second major result in [8] is the uniform asymptotic approximation

$$\frac{1}{n!} m_n(n\alpha; \beta, c)$$
$$= \frac{1}{\Gamma(n\alpha + 1)} n^{n\alpha/2} e^{n\gamma} \left[W_n(\sqrt{n}\,\eta)\, a_0 + \frac{1}{\sqrt{n}} W'(\sqrt{n}\,\eta)\, b_0 + \varepsilon_1 \right], \tag{3.8}$$

where

$$W_n(x) = e^{x^2/4} U(-n\alpha - \tfrac{1}{2}, x) \tag{3.9}$$

and

$$|\varepsilon_1| \leqslant \frac{M_1}{n} |W_n(\sqrt{n}\,\eta)| + \frac{N_1}{n^{3/2}} |W_n'(\sqrt{n}\,\eta)|. \tag{3.10}$$

The constants M_1 and N_1 are independent of n and $\alpha \in [\varepsilon, 1+a]$, and ε can be any fixed small number. The coefficients a_0 and b_0 are given explicitly by

$$a_0 = \frac{u_- h(u_+) - u_+ h(u_-)}{u_- - u_+}, \tag{3.11}$$

$$b_0 = \frac{h(u_-) - h(u_+)}{u_- - u_+}. \tag{3.12}$$

In this section, we shall show that the approximation (3.8) actually holds uniformly for $0 < \alpha \leqslant 1+a$; i.e., the constants M_1 and N_1 in (3.10) are independent of $\alpha \in (0, 1+a]$. First, we need the following result.

LEMMA 2. *Let (η, γ) be the unique solution of the system of nonlinear equations (3.5) for $0 < \alpha \leqslant 1+a$. As $\alpha \to 0^+$, we have*

$$\eta = -(-2 \log c + 2\alpha \log \alpha)^{1/2} + O(\alpha) \tag{3.13}$$

and

$$\gamma = \alpha[\log(1-c) - \log c - \tfrac{1}{2} \log 2 - \tfrac{1}{2} \log(-\log c)] + O(\alpha^2). \tag{3.14}$$

Proof. From (3.3), we have

$$w_+ = 1 - \alpha - \frac{c\alpha^2}{1-c} + O(\alpha^3)$$

and

$$w_- = c + c\alpha + \frac{c\alpha^2}{1-c} + O(\alpha^3).$$

Substituting these into (3.5) yields

$$\begin{aligned} -\alpha\pi i + \alpha\left(1 + \log\frac{1-c}{c}\right) - \alpha \log \alpha + O(\alpha^2) \\ = -\alpha \log u_+ + \eta u_+ - \frac{u_+^2}{2} + \gamma \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} \alpha\pi i - \log c - \alpha[1 + \log(1-c)] + \alpha \log \alpha + O(\alpha^2) \\ = -\alpha \log u_- + \eta u_- - \frac{u_-^2}{2} + \gamma. \end{aligned}$$

Subtracting the last two equations, we get

$$\log c - 2\alpha \log \alpha + \alpha \left[2 + \log \frac{(1-c)^2}{c} \right] + O(\alpha^2)$$
$$= -\alpha \log \frac{\eta + \sqrt{\eta^2 - 4\alpha}}{\eta - \sqrt{\eta^2 - 4\alpha}} + \frac{1}{2} \eta \sqrt{\eta^2 - 4\alpha}. \tag{3.16}$$

We claim that η is bounded away from zero as α approaches the origin. We shall prove this by contradiction, and hence assume that there exists a sequence of positive numbers α_n tending to zero such that its corresponding values η_n satisfy

$$-\frac{1}{n} < \eta_n < 0. \tag{3.17}$$

Since $0 < \alpha < \alpha_-$ corresponds to $-\infty < \eta < -2\sqrt{\alpha}$, (3.17) implies

$$\alpha_n < \frac{1}{4n^2}. \tag{3.18}$$

From (3.16), we have

$$\log c + O(\alpha_n \log \alpha_n)$$
$$= \alpha_n \log(4\alpha_n) - 2\alpha_n \log(-\eta_n - \sqrt{\eta_n^2 - 4\alpha_n}) + \tfrac{1}{2}\eta_n \sqrt{\eta_n^2 - 4\alpha_n}. \tag{3.19}$$

For sufficiently large n, the left-hand side of (3.19) is in absolute value greater than $\frac{1}{2}|\log c|$. On the other hand, by (3.17) the right-hand side of (3.19) is dominated by

$$\alpha_n |\log(4\alpha_n)| + 2\alpha_n |\log(-\eta_n - \sqrt{\eta_n^2 - 4\alpha_n})| + \frac{1}{2n^2}.$$

Therefore, $\alpha_n |\log(-\eta_n - \sqrt{\eta_n^2 - 4\alpha_n})|$ is bounded away from zero as $n \to \infty$; that is, there exists a constant $K > 0$ such that

$$|\log(-\eta_n - \sqrt{\eta_n^2 - 4\alpha_n})| \geqslant \frac{K}{\alpha_n} \geqslant 4Kn^2; \qquad \text{see (3.18)}. \tag{3.20}$$

We shall now show that this is impossible. First we note that

$$|\log(-\eta_n - \sqrt{\eta_n^2 - 4\alpha_n})| \leqslant |\log(-\eta_n)| + \left| \log\left(1 - \sqrt{1 - \frac{4\alpha_n}{\eta_n^2}}\right) \right|. \tag{3.21}$$

Next we observe that since $\eta_n < -2\sqrt{\alpha_n}$, we always have $\alpha_n < (1/4)\,\eta_n^2$ and hence $\log \alpha_n < 2\log(-\eta_n)$, i.e.,

$$|\log \alpha_n| > 2\,|\log(-\eta_n)|. \tag{3.22}$$

If $\alpha_n/\eta_n^2 \nrightarrow 0$, then the second term on the right-hand side of (3.21) is bounded as $n \to \infty$. From (3.20) and (3.21), it follows that

$$\frac{K}{\alpha_n} \leqslant \frac{1}{2}\,|\log \alpha_n| + O(1),$$

which is impossible. If $\alpha_n/\eta_n^2 \to 0$, then the second term on the right-hand side of (3.21) is equal to

$$\log\left(\frac{2\alpha_n}{\eta_n^2} + \frac{2\alpha_n^2}{\eta_n^4} + \cdots\right) = \log 2 + \log \alpha_n - 2\log(-\eta_n)$$
$$+ \log\left[1 + \frac{\alpha_n}{\eta_n^2} + O\left(\frac{\alpha_n^2}{\eta_n^3}\right)\right].$$

Thus, from (3.21) and (3.22), we obtain

$$\log(-\eta_n - \sqrt{\eta_n^2 - 4\alpha_n}) = O(\log \alpha_n),$$

which again contradicts (3.20). Therefore the assumption that there exists a sequence η_n tending to zero (see (3.17)) is incorrect, and our claim is established. Let d be the negative number such that $-\infty < \eta < d$ for all sufficiently small $\alpha > 0$. Equation (3.16) then gives

$$\log c - 2\alpha \log \alpha + \alpha\left[2 + \log \frac{(1-c)^2}{c}\right] + O(\alpha^2)$$
$$= -\alpha \log \alpha + 2\alpha \log(-\eta) - \frac{\eta^2}{2} + \alpha + O(\alpha^2);$$

see the argument for (3.19). This, in turn, yields

$$\eta = -(-2\log c)^{1/2} + O(\alpha \log \alpha) \qquad \text{as} \quad \alpha \to 0^+,$$

or, more precisely, (3.13). The asymptotic formula (3.14) is obtained by substituting (3.13) into (3.15). This completes the proof of Lemma 2. ∎

We now return to the approximation (3.8). To show that (3.8) in fact holds uniformly for $\alpha \in (0, 1+a]$, we must show that the constants M_1 and N_1 in (3.10) are independent of α in $(0, 1+a]$. Since (3.10) has already been established for $\alpha \in [\varepsilon, 1+a]$, $\varepsilon > 0$, we need consider only the case when α is sufficiently small. Let us divide our discussion into two separate case: (i) $\alpha = o(1)$ but $n\alpha$ is unbounded, and (ii) $n\alpha$ is bounded.

In case (i), $n^{2/3}(\eta^2 - 4\alpha)$ is unbounded, as $n \to \infty$, by virtue of (3.13). In case (ii), we also have $\alpha = o(1)$ and, as a consequence, η is bounded away from 0 and $n^{2/3}(\eta^2 - 4\alpha) \to \infty$. Hence, both cases are subsumed under case (I) in [8, Eq. (6.36)]. Therefore, as in [8, Eq. (6.37)], we obtain

$$\varepsilon_1 \sim \frac{1}{n} \frac{\Gamma(n\alpha + 1)}{\sqrt{2\pi}} n^{-(n\alpha/2) - (1/2)} \cdot (\eta^2 - 4\alpha)^{-1/4} \cdot e^{n((\eta^2/4) + (\alpha/2))}$$
$$\times \Bigg\{ e^{n(-\alpha \log((-\eta + \sqrt{\eta^2 - 4\alpha})/2) - (\eta/4)\sqrt{\eta^2 - 4\alpha})} [-2 \sin(\alpha n \pi)\, h_1(u_-)]$$
$$\times \left(\frac{-\eta + \sqrt{\eta^2 - 4\alpha}}{2} \right)^{-1/2} + e^{n(-\alpha \log((-\eta - \sqrt{\eta^2 - 4\alpha})/2) + (\eta/4)\sqrt{\eta^2 - 4\alpha})}$$
$$\times [2 \cos(\alpha n \pi)\, h_1(u_+)] \left(\frac{-\eta - \sqrt{\eta^2 - 4\alpha}}{2} \right)^{-1/2} \Bigg\}. \tag{3.23}$$

(The only difference between this result and (6.37) in [8] is that here we have not applied Stirling's formula to $\Gamma(n\alpha + 1)$. This is because $n\alpha$ is bounded in the present case.) Since

$$-\alpha \log \frac{-\eta + \sqrt{\eta^2 - 4\alpha}}{2} - \frac{\eta}{4} \sqrt{\eta^2 - 4\alpha} = -\frac{\eta}{4} \sqrt{\eta^2 - 4\alpha} + O(\alpha)$$

and

$$-\alpha \log \frac{-\eta - \sqrt{\eta^2 - 4\alpha}}{2} + \frac{\eta}{4} \sqrt{\eta^2 - 4\alpha} = \frac{\eta}{4} \sqrt{\eta^2 - 4\alpha} + O(\alpha \log \alpha)$$

as $\alpha \to 0^+$, from (3.23) it can be shown that

$$\varepsilon_1 = O\left(\frac{1}{n} n^{-(n\alpha/2) - (1/2)} e^{(n/2)\,\eta^2} \right). \tag{3.24}$$

On the other hand, we have from [10, p. 133]

$$U(a, -z) = e^{-i\pi(a + (1/2))} z^{-a - (1/2)} e^{-(1/4)\,z^2} [1 + O(|z|^{-2})]$$
$$+ \frac{\sqrt{2\pi}}{\Gamma(a + \frac{1}{2})} z^{a - (1/2)} e^{(1/4)\,z^2} [1 + O(|z|^{-2})].$$

With $a = -n\alpha - \frac{1}{2}$ and $z = -\sqrt{n}\eta$, we obtain from (3.9)

$$W_n(\eta\sqrt{n}) = -\sqrt{\frac{2}{\pi}}\,\Gamma(n\alpha+1)\, n^{-(n\alpha/2)-(1/2)} e^{(n/2)\eta^2}(-\eta)^{-n\alpha-1}$$
$$\times [\sin(n\alpha\pi) + O(n^{n\alpha+(1/2)} e^{-(n/2)\eta^2})]\left[1 + O\left(\frac{1}{n}\right)\right], \qquad (3.25)$$

where use has been made of the identity $\Gamma(x)\,\Gamma(1-x) = \pi/\sin \pi x$. In view of the recurrence relation

$$U'(a, z) = \tfrac{1}{2} z U(a, z) - U(a-1, z),$$

we also have

$$\frac{1}{\sqrt{n}} W_n'(\eta\sqrt{n}) = -\sqrt{\frac{2}{\pi}}\,\Gamma(n\alpha+1)\, n^{-(n\alpha/2)-(1/2)} e^{(n/2)\eta^2}(-\eta)^{-n\alpha}$$
$$\times [\sin(n\alpha\pi) + O(n^{n\alpha+(1/2)} e^{-(n/2)\eta^2})]\left[1 + O\left(\frac{1}{n}\right)\right]. \qquad (3.26)$$

Since $W_n(x)$ and $W_n'(x)$ do not have common zeros and $n\alpha$ is bounded, a combination of (3.24), (3.25), and (3.26) establishes the validity of (3.10), uniformly for α near the origin; i.e., the constants M_1 and N_1 in (3.10) are independent of n and $\alpha \in (0, \varepsilon]$, $\varepsilon > 0$.

4. BEHAVIOR OF THE SMALL ZEROS

Substituting (3.25) and (3.26) into (3.8), we obtain

$$\frac{1}{n!} m_n(n\alpha; \beta, c) = \sqrt{\frac{2}{\pi}}\, n^{-1/2}(-\eta)^{-n\alpha-1} e^{n\gamma + (n/2)\eta^2}$$
$$\times [(-a_0 - \eta b_0) \sin n\alpha\pi + \tilde{\varepsilon}][1 + O(n^{-1})], \qquad (4.1)$$

where

$$\tilde{\varepsilon} = O(n^{n\alpha+(1/2)} e^{-(n/2)\eta^2}). \qquad (4.2)$$

This result holds uniformly in α, as long as $n\alpha$ is bounded. Let $\bar{\alpha}_{n,s}$ denote the sth zero of $m_n(n\alpha; \beta, c)$, counted from the left; see (1.3). An asymptotic approximation for the zeros $\bar{\alpha}_{n,s}$ will be derived from (4.1)–(4.2) by using Lemma 1. Before proceeding, we first recall the following well-known result [12, p. 44].

LEMMA 3. *The zeros of the Meixner polynomial $m_n(x; \beta, c)$ are real, distinct, and lie in* $(0, \infty)$.

Returning to (4.1), we let

$$f(\alpha) = \frac{1}{n!}\left(\frac{n\pi}{2}\right)^{1/2} e^{-n\gamma - (n/2)\eta^2}(-\eta)^{n\alpha+1} \times m_n(n\alpha; \beta, c)(-a_0 - \eta b_0)^{-1}[1 + O(n^{-1})]^{-1}, \tag{4.3}$$

$g(\alpha) = \sin n\alpha\pi$, and

$$\varepsilon(\alpha) = \tilde{\varepsilon} = O(n^{n\alpha + (1/2)} e^{-(n/2)\eta^2}) \tag{4.4}$$

so that (4.1) becomes

$$f(\alpha) = g(\alpha) + \varepsilon(\alpha). \tag{4.5}$$

Since η is negative and bounded away from zero, we have

$$a_0 + \eta b_0 \sim a_0 + \frac{\eta - \sqrt{\eta^2 - 4\alpha}}{2} b_0 = a_0 + b_0 u_- = h(u_-). \tag{4.6}$$

Using the equation preceding (3.8), it can be shown that

$$h(u_-) \sim \frac{-(-2\log c)^{1/2}}{c(1-c)^{\beta}} \left(\frac{\alpha}{1-\alpha}\right)^{1/2}. \tag{4.7}$$

Since $n\alpha$ is bounded, by Lemma 1 we obtain the following anticipated result.

THEOREM 2. *Let $\bar{\alpha}_{n,s}$ be the sth zero of the Meixner polynomial $m_n(n\alpha; \beta, c)$, arranged in ascending order given in* (1.3). *We have*

$$\bar{\alpha}_{n,s} = \frac{s-1}{n} + O(n^{d-(1/2)} e^{-(n/2)\eta^2}), \tag{4.8}$$

as $n \to \infty$, where d is a constant.

It is interesting to note that the first term on the right-hand side of (4.8) is independent of the parameters β and c. However, we should bear in mind that this formula is valid only when n is sufficiently large. Tables I and II show excellent agreement between the numerical and approximate values of $\bar{\alpha}_{n,s}$ when $\beta = 1.25$, $c = 0.25$, and s is small. On the contrary, Table III gives a very poor comparison when $\beta = 5$, $c = 0.75$, and $n = 30$, but the comparison improves when n becomes bigger. Table IV shows that an agreement is reached in this case when $n = 150$.

TABLE I

Values of $\bar{\alpha}_{n,s}$ When $\beta = 1.25$, $c = 0.25$, and $n = 10$

s	Numerical values	Approximate values
1	$0.12610215 \times 10^{-5}$	0
2	0.10016500	0.1
3	0.20368203	0.2

TABLE II

Values of $\bar{\alpha}_{n,s}$ When $\beta = 1.25$, $c = 0.25$, and $n = 20$

s	Numerical values	Approximate values
1	$0.14590754 \times 10^{-11}$	0
2	$0.50000000 \times 10^{-1}$	0.05
3	0.10000013	0.1
4	0.15000677	0.15
5	0.20015030	0.2

TABLE III

Values of $\bar{\alpha}_{n,s}$ When $\beta = 5$, $c = 0.75$, and $n = 30$

s	Numerical values	Approximate values
1	$0.46487151 \times 10^{-2}$	0
2	$0.58457839 \times 10^{-1}$	0.03333333
3	0.13167135	0.06666666

TABLE IV

Values of $\bar{\alpha}_{n,s}$ When $\beta = 5$, $c = 0.75$, and $n = 150$

s	Numerical values	Approximate values
1	$0.35763984 \times 10^{-14}$	0
2	0.00666666	0.00666666
3	0.01333333	0.01333333
4	0.02000000	0.02
5	0.02666683	0.02666666

The result in Theorem 2 can be strengthened to allow s to depend on n. Indeed, it can be shown that for any fixed $0<\varepsilon<\alpha_-$, there exits a positive number a, depending on ε, such that

$$\bar{\alpha}_{n,s}=\frac{s-1}{n}+O(e^{-an}), \qquad \text{as} \quad n\to\infty, \tag{4.9}$$

for $s=1, 2, ..., \gamma+1$, where $\gamma=[n(\alpha_- -\varepsilon)]$.

To see this, we now let s grow with n since (4.9) has already been established in Theorem 2 when s is fixed. First, we recall the results (6.38) and (6.39) given in [8], namely,

$$\begin{aligned} W_n(\sqrt{n}\,\eta) &= n^{n\alpha/2}\alpha^{n\alpha+(1/2)}e^{n((\eta^2/4)-(\alpha/2))}(\eta^2-4\alpha)^{-1/4} \\ &\quad\times\Bigg\{-2(\sin\alpha\pi n)\,e^{n(-\alpha\log((-\eta+\sqrt{\eta^2-4\alpha})/2)-(\eta/4)\sqrt{\eta^2-4\alpha})} \\ &\quad\times\left(\frac{-\eta+\sqrt{\eta^2-4\alpha}}{2}\right)^{-1/2}[1+O(n^{-1})] \\ &\quad+(\cos\alpha\pi n)\,e^{n(-\alpha\log((-\eta-\sqrt{\eta^2-4\alpha})/2)+(\eta/4)\sqrt{\eta^2-4\alpha})} \\ &\quad\times\left(\frac{-\eta-\sqrt{\eta^2-4\alpha}}{2}\right)^{-1/2}[1+O(n^{-1})]\Bigg\} \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} W_n'(\sqrt{n}\,\eta) &= \sqrt{n}\,n^{n\alpha/2}\alpha^{n\alpha+(1/2)}e^{n((\eta^2/4)-(\alpha/2))}(\eta^2-4\alpha)^{-1/4} \\ &\quad\times\Bigg\{2(\sin\alpha\pi n)\,e^{n(-\alpha\log((-\eta+\sqrt{\eta^2-4\alpha})/2)-(\eta/4)\sqrt{\eta^2-4\alpha})} \\ &\quad\times\left(\frac{-\eta+\sqrt{\eta^2-4\alpha}}{2}\right)^{1/2}[1+O(n^{-1})] \\ &\quad-(\cos\alpha\pi n)\,e^{n(-\alpha\log((-\eta-\sqrt{\eta^2-4\alpha})/2)+(\eta/4)\sqrt{\eta^2-4\alpha})} \\ &\quad\times\left(\frac{-\eta-\sqrt{\eta^2-4\alpha}}{2}\right)^{1/2}[1+O(n^{-1})]\Bigg\} \end{aligned} \tag{4.11}$$

as $n\to\infty$; see also [10, p. 157]. These results hold uniformly with respect to α as long as $\alpha=O(1)$ and $n\alpha\to\infty$. (Note that we have now let s grow with n.) Next, we set

$$h^{\pm}(\alpha)\equiv -\alpha\log\frac{-\eta\pm\sqrt{\eta^2-4\alpha}}{2}\mp\frac{\eta}{4}\sqrt{\eta^2-4\alpha}+\alpha\log\sqrt{\alpha}. \tag{4.12}$$

We shall show that for $\alpha \in (0, \alpha_- - \varepsilon]$, there is a positive number ε_0, depending on ε and c, such that

$$h^+(\alpha) > \varepsilon_0 \qquad \text{and} \qquad h^-(\alpha) < -\varepsilon_0. \tag{4.13}$$

If $\alpha = O(1)$, then $\eta \to -\sqrt{-2 \log c}$ by Lemma 2. Hence

$$h^+(\alpha) \sim \frac{\eta^2}{4} \sim -\frac{1}{2} \log c > 0. \tag{4.14}$$

If α is bounded away from zero, then we may without loss of generality assume $\alpha \in [\varepsilon, \alpha_- - \varepsilon]$. By introducing the new variable $t = -\eta/2\sqrt{\alpha}$, we have

$$h^+(\alpha) = \alpha[-\log(t + \sqrt{t^2 - 1}) + t\sqrt{t^2 - 1}] \equiv \alpha k(t). \tag{4.15}$$

For $\varepsilon \leqslant \alpha \leqslant \alpha_- - \varepsilon$, it can be found in the proof of Theorem 2 in [8] that there
exists a positive number δ_0, depending on ε and c, such that $\eta \leqslant -2\sqrt{\alpha} - \delta_0$. Hence, there is a positive number σ, depending on ε and c, such that $t \geqslant 1 + \sigma$. Since $k'(t) = 2\sqrt{t^2 - 1} > 0$, we obtain from (4.15)

$$h^+(\alpha) \geqslant \varepsilon k(t) \geqslant \varepsilon k(1 + \sigma) \equiv \varepsilon_0 \tag{4.16}$$

for $\alpha \in [\varepsilon, \alpha_- - \varepsilon]$. The first inequality in (4.13) now follows from (4.14) and (4.16). In view of the identity $h^-(\alpha) = -h^+(\alpha)$, the second inequality in (4.13) is also proved.

By using (4.13), we obtain from (4.10) and (4.11)

$$\begin{aligned} W_n(\sqrt{n}\,\eta) &= -2n^{n\alpha/2}\alpha^{n\alpha + (1/2)} e^{n((\eta^2/4) - (\alpha/2))} (\eta^2 - 4\alpha)^{-1/4} \left(\frac{-\eta + \sqrt{\eta^2 - 4\alpha}}{2} \right)^{-1/2} \\ &\quad \times [1 + O(n^{-1})]\, e^{n(-\alpha \log((-\eta + \sqrt{\eta^2 - 4\alpha})/2) - (\eta/4)\sqrt{\eta^2 - 4\alpha})} \\ &\quad \times \{\sin n\pi\alpha + O(e^{-2\varepsilon_0 n} \cdot \alpha^{-1/2})\} \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} W'_n(\sqrt{n}\,\eta) &= 2n^{(n\alpha/2) + (1/2)}\alpha^{n\alpha + (1/2)} e^{n((\eta^2/4) - (\alpha/2))} (\eta^2 - 4\alpha)^{-1/4} \left(\frac{-\eta + \sqrt{\eta^2 - 4\alpha}}{2} \right)^{1/2} \\ &\quad \times [1 + O(n^{-1})]\, e^{n(-\alpha \log((-\eta + \sqrt{\eta^2 - 4\alpha})/2) - (\eta/4)\sqrt{\eta^2 - 4\alpha})} \\ &\quad \times \{\sin n\pi\alpha + O(e^{-2\varepsilon_0 n})\}, \end{aligned} \tag{4.18}$$

respectively. Substituting (4.17) and (4.18) into (3.8) gives

$$\begin{aligned}\frac{1}{n!}\, m_n(n\alpha;\beta,c) &\\ &= \frac{2n^{n\alpha}}{\Gamma(n\alpha+1)}\, e^{n\gamma}\alpha^{n\alpha+(1/2)} e^{n((\eta^2/4)-(\alpha/2))}(\eta^2-4\alpha)^{-1/4}\,[1+O(n^{-1})]\\ &\quad \times e^{n(-\alpha\log((-\eta+\sqrt{\eta^2-4\alpha})/2)-(\eta/4)\sqrt{\eta^2-4\alpha})}\left(\frac{-\eta+\sqrt{\eta^2-4\alpha}}{2}\right)^{-1/2}\\ &\quad \times\left[-a_0+b_0\left(\frac{-\eta+\sqrt{\eta^2-4\alpha}}{2}\right)\right]\{\sin n\pi\alpha+O(\alpha^{-1/2}e^{-2\varepsilon_0 n})\}\end{aligned} \tag{4.19}$$

as $n\to\infty$, uniformly for $0<\alpha<\alpha_- -\varepsilon$. In view of (4.6) and (4.7), it is readily seen that (4.19) yields an equation similar to (4.5), thus establishing (4.9).

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